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# A Liapunov Method for the Estimation of Statistical Averages

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## 1. INTRODUCTION

We consider a randomly perturbed dynamical system described by the equation

$$dx/dt = f(x) + G(x)\xi(t), \qquad t \geqslant 0, \tag{1}$$

where $x, f$ are $n$-vectors, $G$ is an $n \times n$ matrix and $\xi(t)$ is $n$-dimensional Gaussian white noise. Such equations arise in control theory [*1*], and the theory of random vibrations [*2*]. In these applications it is of interest to know under what conditions the process

$$X = \{x(t), \qquad t \geqslant 0\}$$

generated by (1) is stable, in the sense that $X$ admits a unique invariant probability distribution. If $X$ is stable then it is often desirable to estimate various stationary averages $\mathscr{E}\{L(x)\}$, when these averages exist.

In a previous paper [*3*] a criterion of Liapunov type was given for stability in the sense described. In the present note a Liapunov criterion (Theorem 3.1) is obtained for the existence (finiteness) of the stationary average $\mathscr{E}\{L(x)\}$ where $L$ is an arbitrary nonnegative function. This result is applied to show that algebraic moments of all orders exist when, in (1), $G$ is bounded and the unperturbed system $dx/dt = f(x)$ is of Lur'e type.

The existence criterion is extended to yield an effective method of calculating an upper bound for $\mathscr{E}\{L(x)\}$ (Theorem 4.1). The method is illustrated by an example from control theory.

## 2. STATEMENT OF THE PROBLEM

We start with a precise version of (1), namely Itô's equation

$$\begin{aligned} dx(t) &= f(x(t))\,dt + G(x(t))\,dw(t), \qquad t \geqslant 0 \\ x(0) &= x_0\,. \end{aligned} \tag{2}$$

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The following assumptions are made with respect to (2):

(i) $x, f$ are vectors in Euclidean $n$-space $E (n \geqslant 2)$ and $G$ is an $n \times n$ matrix.

(ii) $\{w(t); t \geqslant 0\}$ is a Wiener process in $E$

(iii) $x_0$ is a random variable independent of the process $w(t)$.

(iv) There is a constant $c > 0$ such that

$$|f(x) - f(y)| + |G(x) - G(y)| < c\,|x - y|$$

for all $x, y \in E$. (Here $|\cdot|$ denotes Euclidean norm of a vector or matrix.)

(v) There is a constant $\epsilon > 0$ such that

$$y'G(x)G(x)'y \geqslant \epsilon\,|y|^2$$

for all $x, y \in E$. (A prime denotes transpose of a vector or matrix).

Under these assumptions it is known (cf. [3]) that (2), interpreted in the sense of Itô, defines a continuous, strongly Feller process

$$X = \{x(t), \qquad t \geqslant 0\}.$$

The differential generator of $X$ will be denoted by $\mathscr{L}$, where

$$\mathscr{L}[u(x)] = \tfrac{1}{2}\,\mathrm{tr}[G(x)G(x)'u_{xx}(x)] + f(x)'u_x(x) \tag{3}$$

whenever the indicated derivatives exist. (In (3), $u_x$ is the vector of first partial derivatives of $u$ and $u_{xx}$ is the matrix of second partial derivatives.)

In the following we shall always assume that $X$ is *positive* [4]. Under these conditions it is known [4] that there exists a unique invariant probability measure $\mu$ defined on the Borel sets $B \subset E$: that is, if $P$ denotes probability measure on the paths of $X$, and if

$$P(x_0 \in B) = \mu(B)$$

then

$$P(x(t) \in B) = \mu(B), \qquad t > 0.$$

An effective criterion for positivity of $X$ is given in [3].

Let $L(x) \geqslant 0$ be Hölder continuous on the compact subsets of $E$. The main problem is to obtain a sufficient condition that

$$\mathscr{E}\{L(x)\} = \int_E L(x)\mu(dx)$$

be finite. Subsequently we shall describe a method for deriving an upper bound on $\mathscr{E}\{L(x)\}$.

In the following, the terms *smooth*, and *normal domain*, have the same meaning as in [3].

## 3. A Criterion for Existence of $\mathscr{E}\{L(x)\}$

A Liapunov criterion for the existence of $\mathscr{E}\{L(x)\}$ can be derived by arguments very similar to those of [3] and [4]. The result is given in Theorem 3.1. We start with some preliminary lemmas.

Let $D$ be a normal domain with boundary $\Gamma$, and let $\tau_\Gamma$ be the first time $X$ hits $\Gamma$. Let $\mathscr{E}_x$ denote expectation on the paths of $X$ when $x(0) = x \in E$. Since $X$ is positive, $\mathscr{E}_x(\tau_\Gamma) < \infty$, $x \in E - D$.

Lemma 3.1. *Let*

$$u(x) = \mathscr{E}_x \left\{ \int_0^{\tau_\Gamma} L[x(t)]\, dt \right\}$$

*If $u(x_0) < \infty$ for some point $x_0 \in E - \bar{D}$ then $u(x) < \infty$ for all $x \in E - \bar{D}$. Furthermore*

$$\begin{aligned} \mathscr{L}[u(x)] &= -L(x),\ x \in E - \bar{D} \\ u(x) &= 0,\ x \in \Gamma. \end{aligned} \tag{4}$$

*Proof.* The proof closely follows that of Lemma 5.3 of [4]. Let $\{D_n\,;\, n = 1, 2, \ldots\}$ be an increasing sequence of normal domains such that $D \subset D_1$, $x_0 \in D_1 - D$ and $\lim D_n = E (n \to \infty)$. Let $\tau_n$ be the first time $X$ hits the boundary $\Gamma \cup \Gamma_n$ of $D_n - D$, and define

$$u_n(x) = \mathscr{E}_x \left\{ \int_0^{\tau_n} L[x(t)]\, dt \right\}, \qquad x \in \bar{D}_n - D$$

$$u(x) = \mathscr{E}_x \left\{ \int_0^{\tau_\Gamma} L[x(t)]\, dt \right\}, \qquad x \in E - D.$$

Since $X$ is positive (and hence, regular [4] $\tau_n \uparrow \tau_\Gamma$ $(n \to \infty)$ and therefore $u_n(x) \uparrow u(x) (n \to \infty)$. For fixed $m \geqslant 1$

$$u(x) = u_m(x) + \sum_{n=m}^{\infty} [u_{n+1}(x) - u_n(x)], \qquad x \in \bar{D}_m - D, \tag{5}$$

with convergence at least for $x = x_0$. We now use the fact that $u_n(x)$ is the unique smooth solution of

$$\begin{aligned} \mathscr{L}[u_n(x)] &= -L(x),\ x \in D_n - \bar{D}. \\ u_n(x) &= 0,\ x \in \Gamma \cup \Gamma_n\,. \end{aligned}$$

(see e.g. [5], theorem 13.16). Let

$$v_n(x) = u_{n+1}(x) - u_n(x),\ x \in \bar{D}_n - D.$$

Then

$$\mathscr{L}[v_n(x)] = 0,\ x \in D_n - \bar{D}$$

and (since $u_n(x) \geqslant 0$) $v_n(x) \geqslant 0$, $x \in \Gamma \cup \Gamma_n$. By the maximum principle $v_n(x) \geqslant 0$, $x \in D_n - \bar{D}$. It follows that all terms of the series (5) except the first are positive functions for which $\mathscr{L}[v(x)] = 0$, and the series converges for $x = x_0$. From the generalized Harnack inequality [11] it follows that the series converges for all $x \in \bar{D}_m - D$ and $u(x)$ satisfies (4) for $x \in D_m - \bar{D}$. Since $m$ is arbitrary the result follows.

Lemma 3.2. *A necessary and sufficient condition that*

$$\mathscr{E}\{L(x)\} < \infty$$

*is that*

$$\mathscr{E}_x \left\{ \int_0^{\tau_\Gamma} L[x(t)]\, dt \right\} < \infty, \qquad x \in E. \tag{6}$$

*Proof.* We use the construction and notation of [4]. Let $D_1$ be a normal domain with boundary $\Gamma_1$ such that $D \subset D_1$ and $\Gamma \cap \Gamma_1 = \phi$. Let $\tau$ denote the length of a cycle, namely, in obvious notation,

$$\tau = \min\{t : x(t) \in \Gamma \mid x(0) \in \Gamma \text{ and } x(s) \in \Gamma_1 \text{ for some } s,\ 0 < s < t\}.$$

Let $\tilde{\mu}$ be the finite invariant measure (see [4]) induced on the Borel sets of $\Gamma$. Then if $K$ is an arbitrary compact subset of $E$ we have, within a constant of normalization,

$$\mu(K) = \int_\Gamma \tilde{\mu}(dx)\mathscr{E}_x\{\tau^K\}, \tag{7}$$

where

$$\tau^K = \operatorname{meas}\{t : 0 \leqslant t \leqslant \tau,\ x(t) \in K\}.$$

Let $L_n(x)$ be an increasing sequence of simple functions (constructed on compact sets) such that $L_n(x) = 0(|x| > n)$ and $L_n(x) \uparrow L(x) (n \to \infty)$. From (7)

$$\int_E \mu(dx) L_n(x) = \int_\Gamma \tilde{\mu}(dx)\mathscr{E}_x \left\{ \int_0^\tau L_n[x(t)]\, dt \right\}, \qquad n = 1, 2, \ldots;$$

and by monotone convergence

$$\mathscr{E}\{L(x)\} = \int_\Gamma \tilde{\mu}(dx)\mathscr{E}_x \left\{ \int_0^\tau L[x(t)]\, dt \right\}. \tag{8}$$

Let $\tau_1 = \min\{t : x(t) \in \Gamma_1 \mid x(0) \in \Gamma\}$. By the strong Markov property

$$\mathscr{E}_x \left\{ \int_0^{\tau} L[x(t)]\, dt \right\} = \mathscr{E}_x \left\{ \int_0^{\tau_1} L[x(t)]\, dt \right\}$$
$$+ \mathscr{E}_x \left\{ \mathscr{E}_{x(\tau_1)} \left\{ \int_0^{\tau_\Gamma} L[x(t)]\, dt \right\} \right\}, \qquad x \in \Gamma. \tag{9}$$

We will show that each term in the right side of (9) is bounded on $\Gamma$. Let $u(x)$ denote the first term, regarded as a function on $\bar{D}_1$. Then $u$ is the unique smooth solution of $\mathscr{L}[u(x)] = -L(x)$, $x \in D_1$; $u(x) = 0$, $x \in \Gamma_1$. Since $\bar{D}_1$ is compact, $u$ is bounded. Next, let

$$v(y) = \mathscr{E}_y \left\{ \int_0^{\tau_\Gamma} L[x(t)]\, dt \right\}.$$

By (6) and Lemma 3.1, $v(y)$ is smooth, hence bounded on $\Gamma_1$. By the strong Feller property, $\mathscr{E}_x\{v_x(\tau_1)]\}$ is continuous, and therefore bounded on $\Gamma$. It now follows from (8) that $\mathscr{E}\{L(x)\} < \infty$.

Conversely, if (6) fails for some $x \in E - D$ then by Lemma 3.1, (6) fails for all $x \in E - D$, and by (8) and (9), $\mathscr{E}\{L(x)\} = \infty$.

LEMMA 3.3. *If the equation*

$$\mathscr{L}[v(x)] = -L(x),\ x \in E - \bar{D}$$

*has a smooth positive solution $v(x)$ in $E - D$ then*

$$\mathscr{E}_x \left\{ \int_0^{\tau_\Gamma} L[x(t)]\, dt \right\} < \infty, \qquad x \in E - D.$$

*Proof.* Let $\{D_n;\ n = 1, 2, ...\}$ be a sequence of normal domains constructed as in the proof of Lemma 3.1, and let $u_n(x)$ be the corresponding sequence of smooth functions such that

$$\mathscr{L}[u_n(x)] = -L(x),\ x \in D_n - \bar{D}$$
$$u_n(x) = 0,\ x \in \Gamma \cup \Gamma_n .$$

Since $\mathscr{L}[v(x) - u_n(x)] = 0$, $x \in D_n - \bar{D}$, and $v(x) - u_n(x) \geqslant 0$, $x \in \Gamma \cup \Gamma_n$, we have $u_n(x) \leqslant v(x)$, $x \in \bar{D}_n - D$; therefore

$$\mathscr{E}_x \left\{ \int_0^{\tau_\Gamma} L[x(t)]\, dt \right\} = \lim u_n(x)$$
$$\leqslant v(x), \qquad x \in E - D.$$

This completes the proof.

Before stating Theorem 3.1 we introduce a class of real-valued functions $V$, analogous to Liapunov functions, with the following properties.

$P_1$ : $V$ is defined for $x \in \bar{D}_v$, where

$$D_v = \{x : | x | > R\} \qquad (R < \infty \text{ is arbitrary})$$

$P_2$ : $V$ is continuous in $\bar{D}_v$ and is twice continuously differentiable in $D_v$.

$P_3$ : $V(x) \to +\infty$ as $| x | \to \infty$.

THEOREM 3.1. *Let $X$ be positive. If there exists a function $V$ with properties $P_1 - P_3$ and the additional property*

$$\mathscr{L}[V(x)] \leqslant -L(x), \; x \in D_v$$

*then*

$$\mathscr{E}\{L(x)\} < \infty.$$

We remark that if $X$ is positive and $L(x)$ is bounded then $\mathscr{E}\{L(x)\}$ is obviously finite. If $L(x)$ is bounded away from zero for $x \in D_v$ then, by Theorem 2 of [*3*], the existence of $V$ already implies that $X$ is positive.

*Proof.* By Lemmas 3.2 and 3.3 we have that $\mathscr{E}\{L(x)\} < \infty$ if and only if there exists a normal domain $D$ such that the equation

$$\mathscr{L}[u(x)] = -L(x) \tag{10}$$

has a smooth positive solution $u(x)$ defined for $x \in E - D$. Let $D = E - \bar{D}_v$ and define a sequence $\{D_n\}$ of normal domains as in the proof of Lemma 3.1. The remainder of the proof follows that of Lemma 3.3, with $v(x)$ replaced by $V(x)$. By adding a constant to $V$ if necessary we can arrange that $V(x) \geqslant 0$, $x \in \bar{D}_v$. If

$$\mathscr{L}[u_n(x)] = -L(x) \; (x \in D_n - \bar{D}),$$

$$u_n(x) = 0 \; (x \in \Gamma \cup \Gamma_n),$$

then $0 \leqslant u_n(x) \leqslant u_{n+1}(x) \leqslant V(x)$, $x \in \bar{D}_n - D$. It follows by a compactness theorem ([*6*] p. 344) that $\lim u_n(x)$ exists and is a solution of (10) for $x \in E - \bar{D}$.

*Remark.* The proof of Theorem 3.1 remains unchanged if property $P_3$ of $V$ is replaced by

$$P_3' : V(x) \geqslant 0, \; x \in \bar{D}_v .$$

## 4. Estimation of $\mathscr{E}\{L(x)\}$

In this section we assume that $\mathscr{E}\{L(x)\} < \infty$ and derive an upper bound for this quantity. The result is given in Theorem 4.1.

For $x \in E$ and $t > 0$ define

$$u(t, x) = \mathscr{E}_x \left\{ \int_0^t L[x(s)]\, ds \right\}.$$

The next result is stronger than necessary for the subsequent application, but will be derived for its independent interest.

Lemma 4.1. *If $\mathscr{E}\{L(x)\} < \infty$ then for all $x \in E$,*

$$\lim_{t\to\infty} t^{-1}u(t, x) = \mathscr{E}\{L(x)\}. \tag{11}$$

*Proof.* Let $L_n(x) = L(x)$, $|x| \leqslant n$; $L_n(x) = 0$, $|x| > n$ $(n = 1, 2,\ldots)$. By the corollary to Theorem 3.1 of [4],

$$\begin{aligned} \lim_{n\to\infty} \lim_{t\to\infty} t^{-1}\mathscr{E}_x\left[\int_0^t L_n[x(s)]\, ds\right] &= \lim_{n\to\infty} \mathscr{E}\{L_n(x)\} \\ &= \mathscr{E}\{L(x)\}. \end{aligned} \tag{12}$$

Let $P(t, x, B)$ be the transition function of $X$. If $\mu$ is the invariant measure of $X$ then, by repeated applications of Fubini's Theorem,

$$\begin{aligned} \mathscr{E}\left\{\mathscr{E}_x\left\{t^{-1}\int_0^t L_n[x(s)]\, ds\right\}\right\} &= \int_E \mu(dx)t^{-1}\int_0^t \int_E P(s, x, dy)L_n(y)\, ds \\ &= t^{-1}\int_0^t \int_E \mu(dy)L_n(y)\, ds \\ &= \mathscr{E}\{L_n(x)\}. \end{aligned}$$

Passing to the limit $(n \to \infty)$ we have by monotone convergence

$$\mathscr{E}\{t^{-1}u(t, x)\} = \mathscr{E}\{L(x)\}. \tag{13}$$

Now let $\chi_n(x) = 1$, $|x| \geqslant n$; $= 0$, otherwise. Suppose that for some $\epsilon > 0$ there exists a sequence $\{t_\nu\}(t_\nu > 0)$ and a subsequence $n(\nu)$ of positive integers such that

$$\mathscr{E}_x\left\{t_\nu^{-1}\int_0^{t_\nu} \chi_{n(\nu)}[x(s)]L[x(s)]\, ds\right\} > \epsilon, \qquad \nu = 1, 2,\ldots \tag{14}$$

From (13) and (14) it follows that

$$\mathscr{E}\{\chi_{n(\nu)}(x)L(x)\} > \epsilon,\ \nu = 1, 2,\ldots,$$

which contradicts the fact that $\mathscr{E}\{L(x)\} < \infty$. Hence for each fixed $x \in E$,

$$\mathscr{E}_x \left\{ t^{-1} \int_0^t L_n[x(s)]\, ds \right\} \to \mathscr{E}_x \left\{ t^{-1} \int_0^t L[x(s)]\, ds \right\}$$

as $n \to \infty$, uniformly in $t$ for $t$ sufficiently large. We can therefore interchange limits in the left side of (12), and the result (11) follows by monotone convergence.

We consider functions $V$ with properties $\bar{P}_1 - \bar{P}_3$, where these differ from properties $P_1 - P_3$ of Section 3 only in that now we require $D_v = E$.

THEOREM 4.1. *Let $X$ be positive. If there exist a function $V$ with properties $\bar{P}_1 - \bar{P}_3$ and a positive constant $k$ such that*

$$\mathscr{L}[V(x)] \leqslant k - L(x),\ x \in E,$$

*then*

$$\mathscr{E}\{L(x)\} \leqslant k.$$

*Proof.* We first show that $\mathscr{E}\{L(x)\} < \infty$. Indeed if $D$ is a normal domain with boundary $\Gamma$ and if $v(x) = \mathscr{E}_x\{\tau_\Gamma\}$ then $\mathscr{L}[v(x)] = -1$ $(x \in E - D)$ and $v(x) = 0$ $(x \in \Gamma)$. It follows that the function $V(x) + kv(x)$ satisfies the conditions of Theorem 3.1.

Let $D_n = \{x : |\, x\,| < n\}$ and put $\tau_n = \min\{t : |\, x(t)| = n \mid x(0) = x \in D_n\}$. Let $t_n = \min(t, \tau_n)$ and define

$$u_n(t, x) = \mathscr{E}_x \left\{ \int_0^{t_n} L[x(s)]\, ds \right\}$$

$t > 0$, $x \in D_n (n = 1, 2,...)$. Since $\tau_n \uparrow \infty$ $(n \to \infty)$ with probability 1, we have $u_n(t, x) \uparrow u(t, x)$.

We now use the fact that $u_n(t, x)$ is the unique smooth solution of the problem

$$\begin{aligned} \mathscr{L}[u_n(t, x)] - \partial u_n(t, x)/\partial t &= -L(x), \qquad t > 0, \qquad x \in D_n \\ u_n(0, x) &= 0, \qquad x \in D_n \\ u_n(t, x) &= 0, \qquad t > 0, \qquad |\, x\,| = n \end{aligned}$$

(see e.g. [5], Ch. 5). We can assume that $V(x) \geqslant 0$, $x \in E$. If

$$W_n(t, x) = kt + V(x) - u_n(t, x) \qquad (t \geqslant 0, \qquad x \in \bar{D}_n)$$

then

$$\mathscr{L}[W_n(t, x)] - \partial W_n(t, x)/\partial t \leqslant 0;$$

$W_n(0, x) \geqslant 0$; and $W_n(t, x) \geqslant 0$, $|x| = n$. By the maximum principle for parabolic equations $W_n(t, x) \geqslant 0$ ($t \geqslant 0, x \in \bar{D}_n$); that is $u_n(t, x) \leqslant kt + V(x)$; hence

$$u(t, x) \leqslant kt + V(x), \qquad t \geqslant 0, \qquad x \in E.$$

The result now follows from Lemma 4.1.

*Remark.* The proof is unchanged if property $\bar{P}_3$ of $V$ is replaced by

$$\bar{P}_3' : V(x) \geqslant 0, \qquad x \in E.$$

## 5. Applications

### *EXAMPLE 1*

Let $X$ satisfy the Itô equation

$$\begin{aligned} dx &= Fxdt - b\phi(\sigma)dt + G(x)dw \\ \sigma &= c'x. \end{aligned} \tag{15}$$

In (15), $F$ is a constant matrix, $b$ and $c$ are constant $n$-vectors, and $\phi$ is a scalar-valued, in general nonlinear, function of $\sigma$. The nonstochastic differential equation, obtained from (15) by setting $G = 0$, has been studied extensively in connection with the Lur'e problem [*8*].

THEOREM 5.1. *Let the system* (15) *satisfy the following conditions*:

(i) *All the eigenvalues of $F$ have negative real parts*

(ii) $\sigma\phi(\sigma) > 0$ *for all* $|\sigma|$ *sufficiently large*; $\phi(\sigma)$ *is continuously differentiable*; *and* $d\phi(\sigma)/d\sigma$ *is bounded* ($-\infty < \sigma < \infty$).

(iii) *There exist two nonnegative constants* $\alpha$ *and* $\beta$ *such that* $\alpha + \beta > 0$ *and*

$$\operatorname{Re}(\alpha + i\omega\beta)c'(i\omega I - F)^{-1}b > 0$$

*for all real* $\omega$.

(iv) $G(x)$ *satisfies the conditions of section* 2 *and, in addition,* $|G(x)|$ *is bounded for* $x \in E$.

*Then $X$ is positive and*

$$\mathscr{E}\{|x|^\nu\} < \infty$$

*for every* $\nu > 0$.

*Proof.* The positivity of $X$ was proved in [*3*]. To satisfy the conditions of Theorem 3.1 we introduce a function $\tilde{V}(x)$ of the form

$$\tilde{V}(x) = x'Px + \beta \int_0^{c'x} \phi(\sigma)\, d\sigma$$

and define

$$V(x) = \exp(\gamma \tilde{V}(x)),$$

where $\gamma > 0$ will be chosen later. By a result of Meyer [9] there exist positive definite matrices $P$ and $Q$ such that

$$[Fx - b\phi(c'x)]'\tilde{V}_x(x) \leqslant -x'Qx \tag{16}$$

for all $|x|$ sufficiently large. Moreover

$$\tfrac{1}{2}\operatorname{tr}[G(x)G(x)'\tilde{V}_{xx}(x)] = \operatorname{tr}[G(x)G(x)'P] + \tfrac{1}{2}\beta\,|G(x)'c|^2\, d\phi(c'x)/d\sigma \tag{17}$$

Since the right side of (17) is bounded it follows on adding (16) and (17) that, for arbitrary $\delta > 0$,

$$\mathscr{L}[\tilde{V}(x)] \leqslant -(1-\delta)x'Qx \tag{18}$$

for all $|x|$ sufficiently large. Let $\delta \in (0, 1)$ be fixed. Now

$$\begin{aligned}\exp(-\gamma\tilde{V}(x))\mathscr{L}[V(x)] &= \gamma\mathscr{L}[\tilde{V}(x)] + \tfrac{1}{2}\gamma^2\,|G(x)'\tilde{V}_x(x)|^2\\ &= \gamma\mathscr{L}[\tilde{V}(x)] + \tfrac{1}{2}\gamma^2\,|G(x)'[2Px + \beta\phi(c'x)c]|^2\\ &\leqslant -\gamma(1-\delta)x'Qx + \gamma^2 x'Rx\end{aligned} \tag{19}$$

for some positive definite constant matrix $R$. Since $Q$ is positive definite the matrix $(1 - \delta)Q - \gamma R$ is positive definite for $\gamma > 0$ sufficiently small. Then, for $|x|$ sufficiently large

$$\begin{aligned}\mathscr{L}[V(x)] &\leqslant -\exp(\gamma\tilde{V}(x))\\ &\leqslant -|x|^{\nu}.\end{aligned}$$

The result now follows by Theorem 3.1.

*Remark.* It is clear from the proof that, under the conditions of Theorem 5.1, $\mathscr{E}\{L(x)\} < \infty$ provided

$$L(x) = 0[\exp(\theta\,|x|^2)] \qquad (|x| \to \infty)$$

for $\theta > 0$ sufficiently small.

*EXAMPLE 2*

We shall illustrate the application of Theorem 4.1 to the analysis of a simple control system. Suppose

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -x_2 - \psi(x_1 + x_2),\end{aligned} \tag{20}$$

where

$$\psi(y) = \begin{cases} 1, y \geqslant 1 \\ y, |y| \leqslant 1 \\ -1, y \leqslant -1. \end{cases}$$

The null solution $x_1 = x_2 = 0$ is asymptotically stable. If the system is perturbed by Gaussian white noise it is of interest to estimate the mean square error $\mathscr{E}\{x_1{}^2\}$. The prior verification that $X$ is positive will be omitted. Introducing perturbation terms and making the change of variables $x_1 = x$, $x_1 + x_2 = y$, we obtain

$$\begin{aligned} dx &= -(x - y)\,dt + a_{11}\,dw_1 + a_{12}\,dw_2 \\ dy &= -\psi(y)\,dt + a_{21}\,dw_1 + a_{22}\,dw_2 \end{aligned} \tag{21}$$

where $w_1$, $w_2$ are independent 1-dimensional Wiener processes and the coefficients $a_{ij}$ are constants. The differential generator of the $(x, y)$ process is

$$\mathscr{L}[u] = Au_{xx} + 2Bu_{xy} + Cu_{yy} - (x - y)u_x - \psi(y)u_y\,,$$

where

$$\begin{aligned} A &= (a_{11}^2 + a_{12}^2)/2 \\ B &= (a_{11}a_{21} + a_{12}a_{22})/2 \\ C &= (a_{21}^2 + a_{22}^2)/2. \end{aligned}$$

To satisfy condition (v) of Section 2 we assume that $a_{11}a_{22} - a_{12}a_{21} \neq 0$; in the present application such a restriction is clearly not significant.

To estimate $\mathscr{E}\{x^2\}$ we try to construct a positive function $V(x, y)$ with continuous second derivatives such that

$$\mathscr{L}[V(x, y)] \leqslant k - x^2 \qquad ((x, y) \in E)$$

for some positive constant $k$. As a first step we assume that the perturbation terms are absent from (21) and evaluate

$$V^0(x, y) = \int_0^\infty x(t)^2\,dt \qquad (x(0) = x, y(0) = y).$$

The result is

$$\begin{aligned} V^0(x, y) &= x^2/2 + xy/2 + y^2/4, \ |y| \leqslant 1 \\ &= x^2/2 - x + xy - y^2/2 + y^3/3 \\ &\qquad + e^{1-y}(x - y - 1)/2 + 17/12, \qquad y \geqslant 1 \\ &= V^0(-x, -y), \qquad y \leqslant -1. \end{aligned} \tag{22}$$

From (22) we find that $V^0_{xx}$, $V^0_{xy}$ are continuous, but

$$V^0_{yy}(x, 1-0) = \tfrac{1}{2}, \qquad V^0_{yy}(x, 1+0) = 1 + x/2.$$

To achieve the required smoothness replace $V^0$ by $V^1$, where

$$\begin{aligned} V^1(x,y) &= V^0(x,y) - [\tfrac{1}{4}](1+x)(y-1)^2 e^{-\alpha(y-1)}, \qquad y \geqslant 1 \\ &= V^0(x,y), \qquad |y| \leqslant 1 \\ &= V^1(-x,-y), \qquad y \leqslant -1, \end{aligned} \tag{23}$$

where $\alpha > 0$ is arbitrarily large. From (23),

$$\mathscr{L}[V_1] \sim 2C\,|y| - x^2 \qquad (|x| \to \infty, \quad |y| \to \infty)$$

To cancel the term $2C\,|y|$ for large $|y|$, define

$$\begin{aligned} V^{(2)}(x,y) &= V^1(x,y) + C(|y|-1)^2 \exp[-\beta(|y|-1)^{-1}], \qquad |y| \geqslant 1 \\ &= V^0(x,y),\ |y| \leqslant 1 \end{aligned}$$

where $\beta > 0$ is arbitrarily small. Finally, let

$$V(x,y) = (1+\gamma)V^{(2)}(x,y)$$

where $\gamma > 0$ will be chosen later. Then

$$\mathscr{L}[V(x,y)] \leqslant K - x^2 \qquad (x, y \in E) \tag{24}$$

if $K$ is sufficiently large. By straightforward estimation of the individual terms of $\mathscr{L}[V]$ we can obtain a value $k_\gamma$ of $K$ for which (24) is true; we then choose

$$k = \min\{k_\gamma : \gamma > 0\}.$$

Carrying out the estimates for $|y| \leqslant 1$ and $|y| \geqslant 1$ separately, we find

$$\mathscr{E}\{x^2\} < \max(k', k''),$$

where

$$\begin{aligned} k' &= (A + B + C/2)[1 + C(9C^2 + 4D)^{-\frac{1}{2}}] \\ k'' &= (5/2)C^2 + D + (C/2)(9C^2 + 4D)^{\frac{1}{2}} \end{aligned}$$

and

$$D = A + 2B + |B| + 3C/2.$$

To obtain a rough idea of how conservative the bound may be in this case, suppose that $A \simeq 0$, $B \simeq 0$, $C \to \infty$. Then

$$\mathscr{E}\{x^2\} < k'' \sim 4C^2. \tag{25}$$

Analysis of the system (21) based on "statistical linearization" [10] of the nonlinear function $\psi$ yields

$$\mathscr{E}\{x^2\} \simeq (\pi/2)C^2 \qquad (C \to \infty). \tag{26}$$

The qualitative agreement between the results (25) and (26) is due to the special choice of the function $V_0$. We should emphasize that the upper bound (25) was derived rigorously; the estimate (26), although probably reliable, was obtained by a heuristic procedure.

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